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Comparative empirical analysis of flow-weighted transit route networks in R-space and evolution modeling

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Urban public transit system is a typical mixed complex network with dynamic flow, and its evolution should be a process coupling topological structure with flow dynamics, which has received little attention. This paper presents the R-space to make a comparative empirical analysis on Beijing's flow-weighted transit route network (TRN) and we found that both the Beijing's TRNs in the year of 2011 and 2015 exhibit the scale-free properties. As such, we propose an evolution model driven by flow to simulate the development of TRNs with consideration of the passengers' dynamical behaviors triggered by topological change. The model simulates that the evolution of TRN is an iterative process. At each time step, a certain number of new routes are generated driven by travel demands, which leads to dynamical evolution of new routes' flow and triggers perturbation in nearby routes that will further impact the next round of opening new routes. We present the theoretical analysis based on the mean-field theory, as well as the numerical simulation for this model. The results obtained agree well with our empirical analysis results, which indicate that our model can simulate the TRN evolution with scale-free properties for distributions of node's strength and degree. The purpose of this paper is to illustrate the global evolutional mechanism of transit network that will be used to exploit planning and design strategies for real TRNs.

Keywords: Transit route network; scale-free; evolution; flow dynamics.

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1. Introduction

Urban public transit network (PTN) is a complicated giant system, composed of not only routes and stops but also the passenger flow traveling in the network. As the

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topological structure determines the function of networks, the studies on complex properties of PTNs' structures have attracted great interest of many researchers in recent year, for example, the empirical topological studies with the theory of complex network in the Boston subway system,[1,2] transit networks of Beijing,[3] Poland,[4] German and France,[5] as well as railway networks of India[6] and China.[7,8] They found that there is small-world phenomenon[9] or scale-free behavior[10] in degree distributions of most PTNs (a node's degree is defined as the number of neighbors that it is linked to). However, most of researches just focused on the static features of topological structures, while the dynamical characteristics reflected by passengers' travel behaviors are not given adequate consideration. Recently, the studies on weighted PTNs related to passenger flow, to our knowledge, are carried out in two ways. One is modeling the network according to passengers' travel routes, for example, Seoul subway[11] and bus system,[12] and Singapore public transportation system.[13] Lee *et al.*[11] found that the edge weight of Seoul subway system follows a power-law distribution with exponent $\gamma = 0.56$, and Soh *et al.*[13] uncovered that both the edge weight and node's strength also obey the power-law distributions with $\gamma \in [12.5]$ for the Singapore subway and bus network. The other way is studying the flow-weighted PTNs in L-space,[14] which takes both the flow and underlying physical structure into consideration.

These considerable empirical studies attract more attention to the evolution of PTNs. Why do different PTNs' topologies show similar properties of small-world or scale-free? How to simulate the PTNs' evolution to reveal these global features? What role does the passenger flow play in the evolution of topological structure? How the flow and topology interact and impact mutually? In seeking answers to these questions, some researchers have tried to model the evolutional process for PTNs based on network or graph methods. Su *et al.*[15] proposed a game theory model for simulating the evolution of PTNs that experimental results show agreement with the degree distribution of actual PTN. He *et al.*[16] put forward a growth model to simulate the transit network through analyzing the static properties of real PTNs. Besides, some typical models to simulate the PTNs evolution are proposed recently, for instance, the model with self-avoiding random walks,[17,18] the model in P-space[19] and in L-space,[20] the ideal n-depth clique network model,[21] and the space evolution model that could satisfy both the passengers and investors' demands.[20,22] They also used empirical evidence to validate the models. However, for most of studies, the passenger flow distributed in topologies and the dynamic behavior triggered by the topological evolution have not yet been taken into account. Although topology has great significance for networks' function, recent studies showed that connectivity models alone cannot provide sufficient information about the flow performance of real systems.[23–30] For the model considering flow dynamics, a typical one is BBV model[23,24] that the network's growth is driven by traffic flow needs. This model can successfully reproduce the scale-free properties for node's degree, strength and edge weight distributions. However, since PTN is a social-economic system with passengers who travel in topological network, most of current models depend on some unrealistic assumptions that cannot completely reveal the principles and nature of PTN's evolution.

In light of the review, the evolutional mechanism of PTNs with consideration of dynamic flow, as well as the interaction between flow and topology, has still received little attention, despite the key role that the passenger flow plays in topological evolution and network's operation. After all, the development of transportation systems are accompanied by assigning the origin-destination flow across the network and the network's operation is not just a connectivity problem. In order to illuminate the PTNs' overall layout and intrinsic characteristics, it is significant to model the flow-based PTNs at the level of transit lines, which is totally different from the former studies that model the networks based on travel routes or station[11–14] and route's section (i.e., in L-space[20]). Hence, in this paper, we will present the R-space to probe the flow properties of real PTN from the macroscopic viewpoint of transit line layout, and propose a model driven by passenger flow to simulate the evolutional process of transit route network (TRN) with consideration of flow behaviors. The model will be validated by theoretical analysis and numerical simulation, as well as our empirical analysis of real PTNs.

The paper is structured as follows. In the next section, we make a comparative empirical analysis on the complex properties of flow-weighted TRN in Beijing, which are the basis for modeling evolutional process and used to validate the model results. In Sec. 3, a flow-driven evolution model for weighted TRN is proposed, and a theoretical solution for the model is presented with the mean-field theory. In Sec. 4, we make a numerical simulation to study the impacts of flow dynamic behavior on the network's structure. Conclusions and future research are given in Sec. 5.

2. Comparative Empirical Analysis of Flow-Weighted Network for Beijing's Bus Transit System

2.1. *Definition of R-space and network generation*

We define the R-space to measure the weighted complex network of transit route system. R-space is a method to describe the spatial connection of node and edge in transit network. Different with "L-space" or "P-space",[20] the R-space regards the end of a route (i.e., the first or ending stop/terminal) in bus route network as a node, and links an edge between two nodes if they are two ends of a route. The edge weight w_{ij} between node i and node j is represented by the average daily passenger flow P_{ij} of a route. In general, the flow is bi-directionally balanced for a whole day,[31] so we make the assumption that the network is undirected ($w_{ij} = w_{ji}$) by averaging the in and out edge weights. An example of three routes in R-space is shown in Fig. 1. This R-space can be exploited to analyze the overall layout of topological network and flow distribution at the macroscopic level.

Beijing is one of the most populated cities in the world. Nearly 29% of daily trips depend on bus transit in 2015, that is to say, approximate 13 million trips were

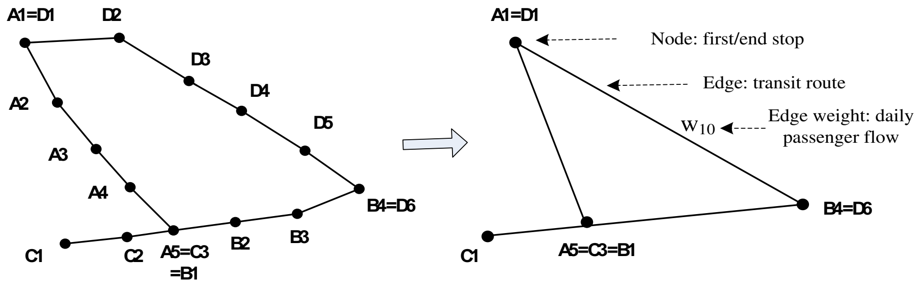


Fig. 1. Transformation of actual public transit routes to flow-weighted TRN in R-space.

made on bus system daily. As a result, Beijing bus network is also one of the most complicated systems in the world. To meet the demand well, every year the number and routing of transit lines are readjusted by Beijing Public Transportation Group Corporation. In order to probe the evolution of transit networks, we utilized the Intelligent Card system to collect the data including routes, stops and passenger flow in weekdays of the year 2011 and 2015, which are composed of 565 and 727 routes, respectively. Correspondingly, two flow-weighted TRNs are constructed with 346 nodes, 386 edges for 2011 and 448 nodes, 488 edges for 2015. In the following, we will make a comparative analysis on the basic complex properties of TRNs in Beijing.

2.2. *Degree, strength and weight distribution*

In TRNs, the degree k_i of node i represents the number of routes that the first or end stop i is linked to Fig. 2 which illustrates the distributions of node's degree for Beijing's weighted TRNs in 2011 [Fig. 2(a)] and 2015 [Fig. 2(b)], respectively. We observe that both the degree distributions of 2011 and 2015 on log–log scales

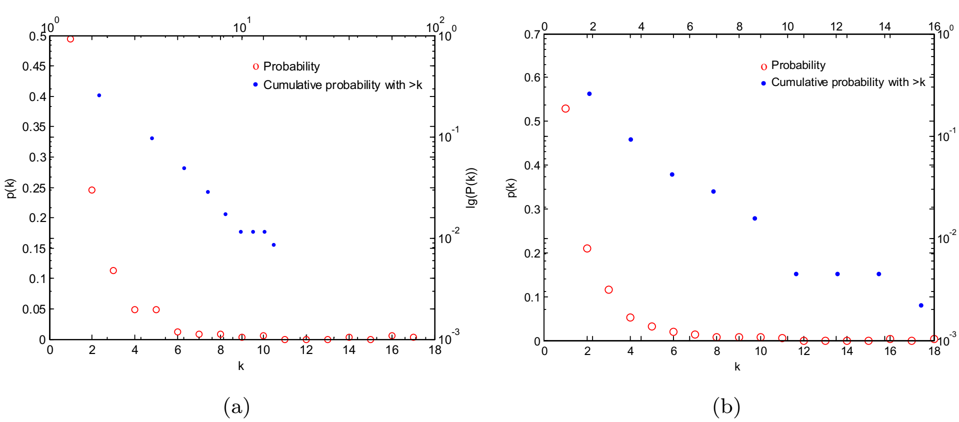
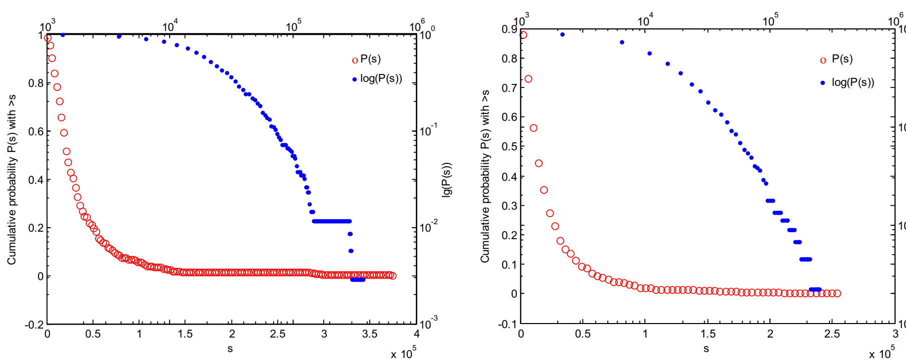


Fig. 2. (Color online) Node's degree distribution and its log–log scale plot for weighted TRN in Beijing. (a) Node's degree distribution in 2011. (b) Node's degree distribution in 2015.

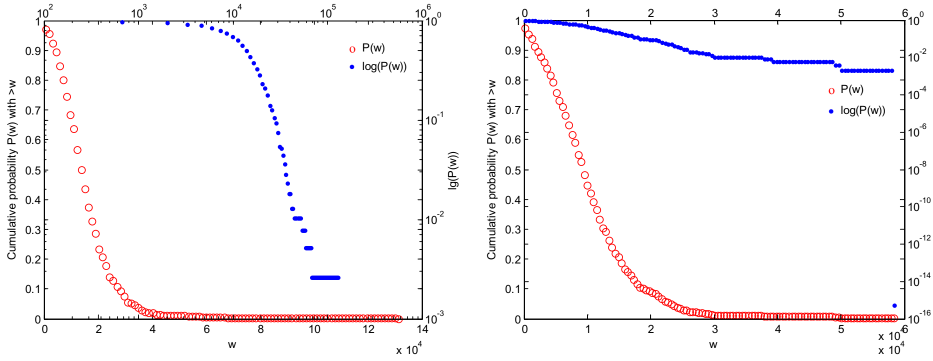
are linear (the blue dots), which is the characteristic signature of power-law.[10] It indicates that in spite of the route adjustment year by year, the degree of TRNs in Beijing keeps the scale-free property that all the distributions follow power-law function $p(k) \sim k^{-\gamma}$. The average degrees are also similar with $\langle k \rangle \approx 2.23$ in 2011 and $\langle k \rangle \approx 2.18$ in 2015. However, the homogeneity in 2015 is slightly enhanced with the increase of power-law exponent $\gamma = 1.565$ compared with $\gamma = 1.458$ for 2011. The scale-free property of degree reveals that the topology of Beijing's TRN in R-space is a hub-and-spoke structure that there are some nodes called hubs possessing much more degrees than other nodes. How hubs form in topological network and what factors influence more routes to connect with these hubs? Traditional theories interpret that nodes with higher degrees tend to grow more edges with other nodes. However, as the travel demand of passenger is one of the key factors to set up bus routes, the travel flow pattern and relationship between flow distribution and topology structure should be investigated to explore if the flow has impact on topologic forming. In following, further study on passenger flow pattern will be presented.

As the edge weight w_{ij} represents the average daily passenger flow of route r_{ij}, the strength s_i of node i, defined as $s_i = \sum_j w_{ij}$,[24] denotes the average daily flow of all routes with a same end i. For the strength distributions (as shown in Fig. 3), their long-tailed behaviors (the red circles) and approximately linear tails on log–log scales (the blue dots) show that they also have the properties of power-law both in 2011 and 2015. It reveals that the flow distribution of nodes is also a hub-and-spoke structure with great heterogeneity that passenger flow handled by each end stop differs significantly. However, contrary to the degree, the heterogeneity for strength distribution is increased from 2011 to 2015 with the exponent $\gamma = 0.9688$ decreasing in 2015, compared with $\gamma = 1.564$ in 2011. Furthermore, the level of heterogeneity for strength is quite higher than that of degree in 2015, while 2011 keeps the same. This can be explained further by the relationship between strength



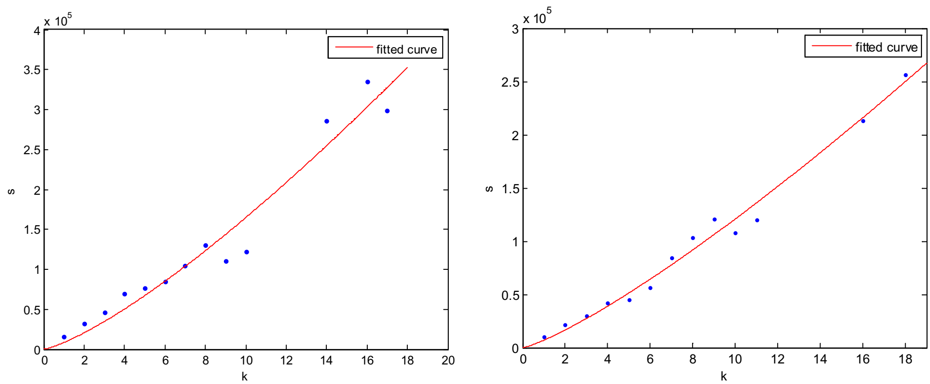
(a) (b)

Fig. 3. (Color online) Node's strength distribution and its log–log scale plot for weighted TRN in Beijing. (a) Node's strength distribution in 2011. (b) Node's strength distribution in 2015.



(a) (b)

Fig. 4. Edge weight distribution for weighted transit route network in Beijing. (a) Edge weight distribution and its log-log scale plot in 2011. (b) Edge weight distribution and its semi-log scale plot in 2015.



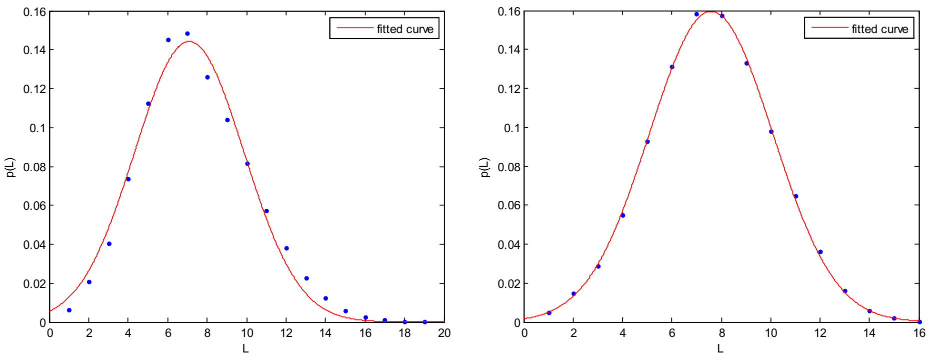
(a) $s = 8273k^{1.298}$ in 2011 (b) $s = 7033k^{1.235}$ in 2015

Fig. 5. The strength spectrums as a function of degree for weighted TRN in Beijing.

and degree. As shown in Fig. 4, although the edge weight distribution in 2015 is less heterogeneous than 2011 because of the exponential distribution in 2015 (indicated by its linearity on semi-log scale), whereas the power-law distribution in 2011, the strength spectrums (as shown in Fig. 5) shows that both the strengths in these two years increase exponentially with degree. Therefore, it is the underlying topology to guide the flow distribution and the topological heterogeneity make it form the scale-free structure; at the same time, the exponential growing of strength enhances the heterogeneity of flow distribution.

2.3. *Path length*

The path length d_{ij} for two nodes i, j in a network is the number of edges on the shortest path between i and j. For the TRN in R-space, the path length represents



(a) In 2011 (b) In 2015

Fig. 6. The path length distribution for weighted TRN in Beijing.

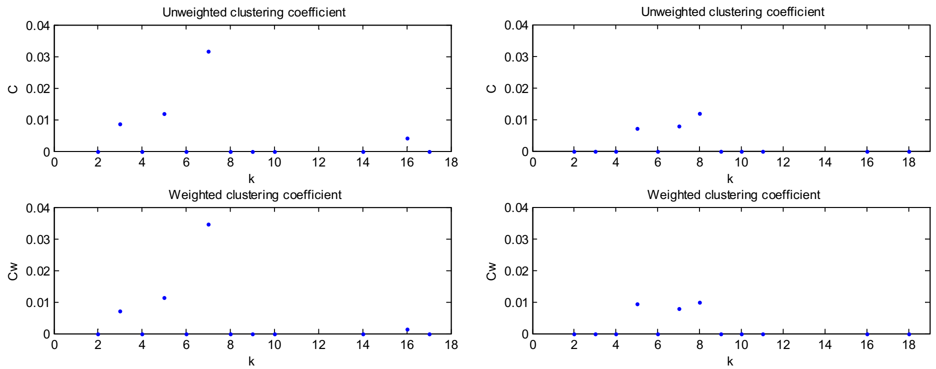
the minimum transit routes to ride from one terminal to the other. The average path length (APL) L is the mean of all d_{ij} for any two nodes in a network:

$$L = \frac{2}{N(N-1)} \sum_{i \geq j} d_{ij}, \qquad (1)$$

where, N is the number of nodes in the network. As can be seen from Fig. 6, both the path length in 2011 and 2015 follow the normal distribution and two TRNs have quite shorter APL, compared with the network diameter D (means the longest path), i.e., $\langle L \rangle = 7.41$ relative to $D = 19$ in 2011 and $\langle L \rangle = 7.62$ to $D = 16$ in 2015.

2.4. *Clustering coefficient*

To further verify the scale-free property of TRNs, here, the unweighted and weighted clustering coefficients are analyzed. The clustering coefficient is defined as follows.[9] Suppose that node i has k_i neighbors; then at most $k_i(k_i - 1)/2$ edges can exist between them. Let E_i denote the number of actually existing edges between k_i neighbors, then the clustering coefficient c_i of node i is the ratio of E_i to $k_i(k_i - 1)/2$. Define C as the average of c_i over all i. For the weighted clustering coefficient c_i^w,[32] with considering the weights of edges linked between i and its neighbors, it can reflect the importance of i's neighbors more completely. In TRNs, the stop's clustering coefficient is a measure of the degree to which transit routes in the network tend to cluster together. As shown in Fig. 7, in 2011, average unweighted clustering coefficient $\langle C \rangle$ is 0.00364, weighted $\langle C^w \rangle$ is 0.00331, and in 2015, $\langle C \rangle$ is 0.00087, weighted $\langle C^w \rangle$ is 0.00099. It shows that both the unweighted and weighted clustering coefficients in 2011 and 2015 are very small, indicating that there is no significant interconnected communities between nodes whether the edge weights are taken into account or not. The small clustering, together with shorter average path length, further indicates that the scale-free features exist in the TRNs of Beijing, as suggested by Barabási and Albert.[10] According to Fig. 7, in 2011, for the routes



(a) In 2011

(b) In 2015

Fig. 7. The unweighted and weighted clustering coefficient distributions for weighted TRN in Beijing.

with one ends' degree being 7, the other ends tend to cluster together compared with that of 2015.

In sum, since the topology and strength distributions are scale-free and heterogeneous, while edge weight is more homogenous, we can conjecture an evolutional mechanism of TRNs like this: as the passengers' travel behavior on transit routes exhibit the complicated characteristics composing of preferentially choosing[10] (according to scale-free property of strength distribution) and randomly evolving[9] (according to small-world phenomenon of edge weight distribution) together, it is the heterogeneity of underlying topology that leads the flow to tend to assemble or distribute in one or some specified spatial stops, which makes their strength preferentially grow; conversely, in order to meet the traffic demands, more new routes will be extended from these preferential stops; consequently, this travel distribution will further enhance the preferential growth of nodes' degree. The interaction between spatial stop location and travel behavior finally leads the TRN's topology and flow distribution structure to macroscopically present the similar scale-free properties after a long-term evolutional process.

3. Evolution Model Driven by Flow for Weighted TRNs

In light of the empirical analysis results, we found that in essence, both the passenger's travel behavior and topological structure contribute to the evolution of TRNs together. Here, we propose an evolution model driven by passengers' travel demands for weighted TRNs with consideration of flow dynamic behavior triggered by topological change.

3.1. *Modeling*

Initial state: the original network is defined as an existing TRN with m_0 stops/nodes and e_0 edges. The original edge weight $w_{ij}(0)$ denotes the total volume of passengers' demands on transit routes between two end stops i and j. The initial strength of node i is $s_i = \sum_j w_{ij}(0)$. In each time step, the routes will be generated according to the following two regulations at the same time:

(1) **Regulation 1:** Generating routes by linking existing stops with probability p. It means that the new routes are set up through extending from existing stops with a probability p. There are two steps to achieve Regulation 1:

Step 1. Generating a route. A pair of nodes is selected from the existing network and a new route is set up by linking these two selected nodes based on the principle of flow-driven. As the transit service is always provided for meeting the demands of passengers, i.e., the pathway of a transit route must be consistent with the flow direction, a route should be extended from a node with larger strength in which more trips are generated or attracted. Thus, a node will be selected as one end of a new route with the probability $\prod_{\text{new}\to i} = \frac{s_i}{\sum_j s_j}$. For the other end of a route, we assume that it is randomly selected from existing stops. The end of a route is not necessarily the stop with larger flow due to the constraints, such as locations of bus terminals, passengers' convenience of accessing. These two ends are linked to form a new route, i.e., an edge. As choosing the node with the most passenger flow as the starting or ending stop is reasonable in real world, the nodes are allowed to be reselected. Thus, m edges are linked between m pairs of nodes in turn.

Step 2. Dynamically evolving of passenger flow. (i) The weight of a new link evolves. When a new route r_{ij} is added to the network, if there is no link between two chosen nodes i, j, an edge with w weight will be linked between i and j; if there exists one, its weight will be added additional w. (ii) The nearby flow perturbation triggered by a new route. When a new transit service is provided, it will lead to a flow perturbation δ in nearby routes, which could be negative when the new route attracts more passengers from nearby routes, or positive when more car users are attracted to ride buses because of the improvement of transit network connectivity. Provided δ_i, δ_j are the additional flow for nodes i, j triggered by a new link w_{ij}, the strength of i, j will be adjusted to be $s_i \to s_i + \delta_i + w$, $s_j \to s_j + \delta_j + w$ respectively. For simplicity, only the neighboring routes linked to i or j are triggered to generate flow perturbation, namely, the weight between i and its adjacent node k, $k \in v(i)$, or j and adjacent n, $n \in v(j)$, are readjusted to share the flow perturbation δ_i or δ_j according to their original weight. The flow evolution will follow the regulation shown in Fig. 8. For node i: $w_{ik} \to w_{ik} + \Delta w_{ik}$, $\Delta w_{ik} = \delta_i \frac{w_{ik}}{s_i}$. Likewise, for node j: $w_{jn} \to w_{jn} + \Delta w_{jn}$ and $\Delta w_{jn} = \delta_j \frac{w_{jn}}{s_j}$.

Repeat Steps 1 and 2, m ($m \le m_0(m_0 - 1)/2$) pairs of stops will be selected in turn from existing network, and m denotes the number of routes to be set up at each time step. When the first pair of stops is selected, the flow distribution

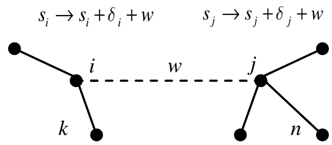


Fig. 8. The rule of flow evolution triggered by linking a new route between node i and j.

will change in terms of aforementioned flow evolution principle in Step 2, then the second pair of stops will continue to be selected from the new network. According to this order, finally m pairs of stops are selected. Thus, m edges are linked between m pairs of nodes to form m routes. Considering the rate of route increase from the existing network, the probability p is introduced to control the number of new route generation.

(2) **Regulation 2:** Generating routes by adding one new terminal with probability $1-p$. It denotes that m new routes are extended from a new terminal with a probability $1-p$. In this regulation, m nodes will be selected from the existing network in turn according to the flow attraction principle, i.e., be selected with probability $\prod_{\text{new}\to i} = \frac{s_i}{\sum_j s_j}$, and a single new node will be added into the network with a probability $1-p$. Then, m routes are generated by linking m selected nodes to the new node. The weights w will be added to these m routes, respectively. Likewise, the strength of a selected node i will be readjusted as $s_i \to s_i + \delta_i + w$, and weights between node i and its neighboring nodes will also be adjusted according to the rule of Fig. 8.

3.2. *Theoretical analysis*

Generally, there are three theoretical methods to solve the distribution of scale-free network: the mean-field,[33] the master equation[34] and the rate equation.[35] The results solved by these methods are basically consistent. In the following, we will use the mean-field theory to analyze the feature of strength distribution generated by above evolution model. For simplicity, given $w = w_0$. In terms of Regulation 1 in Sec. 3.1, we can reduce the rate equation of weight change in the network as follows:

$$\begin{aligned}\frac{\partial w_{ij}}{\partial t} &= \frac{mw_0}{N_t} \times \frac{s_i}{\sum_l s_l} + \frac{mw_0}{N_t} \times \frac{s_j}{\sum_l s_l} + \left(m\frac{s_i}{\sum_l s_l} + \frac{m}{N_t}\right) \\ &\quad \times \delta_i \frac{w_{ij}}{s_i} + \left(m\frac{s_j}{\sum_l s_l} + \frac{m}{N_t}\right) \times \delta_j \frac{w_{ij}}{s_j} \\ &= \frac{mw_0}{N_t}\frac{s_i + s_j}{\sum_l s_l} + \left(m\frac{s_i}{\sum_l s_l} + \frac{m}{N_t}\right) \times \delta_i \frac{w_{ij}}{s_i} + \left(m\frac{s_j}{\sum_l s_l} + \frac{m}{N_t}\right) \times \delta_j \frac{w_{ij}}{s_j}. \quad (2)\end{aligned}$$

The first item of the right of Eq. (2) indicates that for the nodes i and j which we consider to select, node i is selected with a preferential probability $\prod_{\text{new}\to i} = \frac{s_i}{\sum_j s_j}$, while node j with a probability $\frac{1}{N_t-1}$ as the present network scale is N_t. Note that when N_t is very large, $\frac{1}{N_t-1} \approx \frac{1}{N_t}$. Every time a successful link makes the weight w_{ij} increase w_0, and there are m times to choose a pair of nodes. For the second item, it is the case that node i is randomly selected whereas node j preferentially. The third item denotes that once node i is selected with a random or preferential probability, the flow perturbation around node i is triggered; *vice versa*, the last item is the case that node j is selected.

In terms of Regulation 2 in the model, we obtain the rate equation of weight change as follows:

$$\frac{\partial w_{ij}}{\partial t} = m\frac{s_i}{\sum_l s_l} \times \delta_i \frac{w_{ij}}{s_i} + m\frac{s_j}{\sum_l s_l} \times \delta_j \frac{w_{ij}}{s_j}. \tag{3}$$

According to the mean-field theory, the change rate of s_i in network should be equal to the total increased strength triggered together by Regulations 1 and 2. Thus, the change rate of s_i is:

$$\begin{aligned}\frac{\partial s_i}{\partial t} &= p \sum_{j\in\nu(i)} \begin{bmatrix} \frac{mw_0}{N_t}\frac{s_i+s_j}{\sum_l s_l} + \left(m\frac{s_i}{\sum_l s_l} + \frac{m}{N_t}\right) \times \delta_i \frac{w_{ij}}{s_i} \\ + \left(m\frac{s_j}{\sum_l s_l} + \frac{m}{N_t}\right) \times \delta_j \frac{w_{ij}}{s_j} \end{bmatrix} \\ &\quad + (1-p)m \sum_{j\in\nu(i)} \left[\frac{s_i}{\sum_l s_l}\left(w_0 + \delta_i \frac{w_{ij}}{s_i}\right) + \frac{s_j}{\sum_l s_l}\delta_j \frac{w_{ij}}{s_j}\right]. \end{aligned} \tag{4}$$

For simplicity, given that $\delta_i = \delta_j = \delta$. Based on the evolutional regulations, the network scale N is the function of time, $N(t) = m_0 + (1-p)t$. When t is large enough, the impact of m_0 on the result can be neglected, i.e., $N(t) = (1-p)t$. Therefore, Eq. (4) can be expressed as:

$$\begin{aligned}\frac{\partial s_i}{\partial t} &\approx m\frac{s_i}{\sum_l s_l}(w_0 + \delta - pw_0) + \frac{pm}{N}(w_0+\delta) + m\delta \sum_{j\in\nu(i)} \left(\frac{s_j}{\sum_l s_l} \times \frac{w_{ij}}{s_j}\right) \\ &\quad + \frac{pm\delta}{N}\sum_{j\in\nu(i)} \frac{w_{ij}}{s_j}. \end{aligned} \tag{5}$$

As the total change of strength in network triggered by the links between old nodes and the new node with old one is $2[p(w_0+2\delta) + (1-p)(w_0+\delta)] = 2(w_0 + p\delta + \delta)$, then $\sum_{i=1}^{t} s_i(t) = 2m(w_0 + p\delta + \delta)t$. Based on the boundary condition $s_i(t=i) = mw_0$, we obtain the function of strength $s_i(t)$ for node i as follows:

$$s_i(t) = \frac{mAw_0 + B}{A} * \left(\frac{t}{i}\right)^A - \frac{B}{A}, \tag{6}$$

where $A = \frac{w_0+2\delta-pw_0}{2(w_0+p\delta+\delta)}$, $B = \frac{pm(w_0+2\delta)}{1-p}$. According to Regulation 2 for adding a new node, we get the probability distribution of t_i:

$$P(t_i) = \frac{1}{m_0+t}. \tag{7}$$

As the probability that strength $s_i(t)$ for node i is less than S, is as follows:

$$P(s_i(t) < s) = P\left(t_i > t * \left(\frac{mAw_0 + B}{sA + B}\right)^{\frac{1}{A}}\right) = 1 - \frac{t}{m_0 + t * \left(\frac{mw_0 + B/A}{s + B/A}\right)^{\frac{1}{A}}} \tag{8}$$

Therefore, we obtain the probability density function $P(s)$ of strength $s_i(t)$ for node i is:

$$P(s) = \frac{\partial P(s_i < s)}{\partial s} = \frac{t}{m_0 + t} * \frac{(mw_0 + B/A)^{\frac{1}{A}}}{A * (s + B/A)^{1+\frac{1}{A}}} \tag{9}$$

When $t \longrightarrow \infty$, the strength distribution is approximately as follows:

$$P(s) \approx \frac{(mw_0 + B/A)^{\frac{1}{A}}}{A * (s + B/A)^{1+\frac{1}{A}}} \tag{10}$$

According to Eq. (10), we find that when the network's scale is large enough, the node's strength in the evolution model follows a power-law distribution with an exponent $\gamma = 1 + \frac{1}{A} = \frac{\delta(2p+4)+w_0(3-p)}{w_0+2\delta-pw_0}$, which agrees well with our empirical analysis result in Sec. 2.2. Compared with BBV model,[23] it corresponds to just shifting a constant B/A. Obviously, when $p = 0$, the network is completely growing and if $w_0 = 1$, $\delta = 0$, then $\gamma = 3$; if $\delta \to +\infty$, then $\gamma \to 2$. Therefore, the strength of model follows atypical scale-free distribution with $\gamma \in (2, 3]$. At this point, it corresponds to the classical BA model.[10]

In addition, according to the discovery of BBV model,[23] there is a linear relation between node's degree k_i and strength s_i, so the probability density distribution of k_i is approximately as follows:

$$P(k) \propto (k + \alpha)^{-\gamma}. \tag{11}$$

Namely, it has the same power-law exponents as the $P(s)$, but just shifts a constant α.

Table 1. Summary of the symbols.

No.	Symbol	Denotation
1	k_i	The degree of node i
2	w_{ij}	The weight of edge between node i and j
3	s_i	The strength of node i, $s_i = \sum_j w_{ij}$
4	m	The number of new routes
5	m_0	m_0 nodes in existing network
6	e_0	e_0 edges in existing network
7	$w_{ij}(0)$	The original edge weight $w_{ij}(0)$
8	p	The probability of new routes generated from existing nodes
9	$1 - p$	The probability of new routes generated by adding a new node to existing network
10	δ_i	The additional flow for nodes i triggered by a new edge e_{ij}
11	N_t	Total number of nodes in the present network
12	γ	The exponent of power-law distribution

In order to understand above equations more clearly, here we summarize all the symbols as shown in Table 1.

4. Numerical Simulation and Analysis on Impacts of Flow Perturbation

According to Eq. (10), when $p = 0$, it means to extend new routes through establishing a new terminal, and then $\gamma = 2 + \frac{w_0}{w_0+2\delta}$; while $p = 1$, $\gamma = 3 + \frac{w_0}{\delta}$, it denotes to open new routes based on existing stops. In this section, we will study the impacts of flow perturbation δ on network's flow distribution and topological structure through numerical simulation when $p \in (0, 1)$. We set an initial fully-connected network with $m_0 = 10$ nodes and random weights $w_{ij}(0) \in (0, 1)$ as travel demand on links between pairs of nodes. The evolution process is alternated 1000 times. Assume $w_0 = 1$ for all following cases.

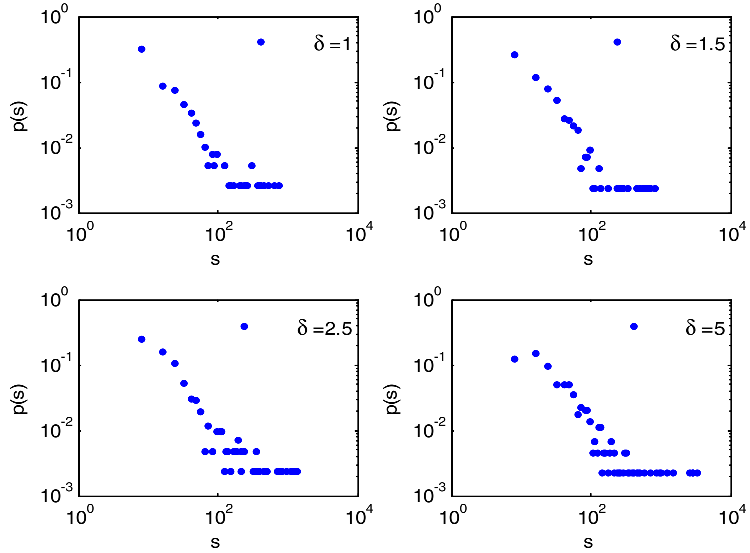
4.1. *When flow perturbation* $\delta \geq 0$

(1) When $\delta > w_0$, the new route should be arterial. The simulation results are shown as Fig. 9 when $p = 0.2$ or $p = 0.8$. We find that whatever p and δ are, the strength distribution plot is nearly linear on log–log scale, which is consistent with our result of theoretical analysis, and similar with our empirical results. But with the increment of δ, e.g., $\delta = 5$, there is a slightly bowing phenomenon of the head for distribution plots that is more similar with real PTNs,[20] indicating that the heterogeneity has decreased in the network; while when $\delta \to +\infty$, $\gamma \in (2, 3)$, it is a typical scale-free network. In addition, when p decreases, the exponent γ becomes smaller, i.e., $\gamma_{p=0.2} < \gamma_{p=0.8}$, revealing that when the new routes are opened mainly by establishing new terminals, the network will become more heterogeneous.

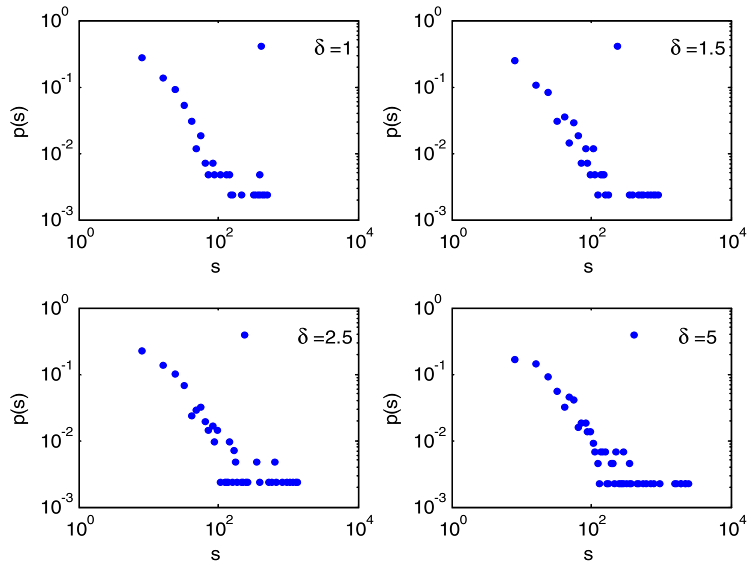
    In summary, when $\delta > w_0$, it reveals that the new route generates a sort of multiplicative effect[23] that is bursting great traffic on neighbors, so it corresponds to an arterial route that attracts great flow of not only transit passengers but also those diverted from other transportation modes, for instance, car user trips. Since such routes have great impacts on neighbors' flow, more attention should be paid to them before opening or in operation.

(2) When $\delta \approx w_0$, the new route should be feeder or community service. When $\delta \approx w_0$, $\gamma \in (2\frac{1}{3}, 4)$, $P(s)$ also follows a power-law distribution (shown in Fig. 9). As $\delta \approx w_0$, it indicates that the passenger flow generated by the new route is totally assigned to nearby routes, and to get to the destinations, passengers need to transfer between the new route r_{ij} and neighbors. Accordingly, the terminal i or j mainly plays a role of transfer station, and the new route generally provides feeder or community service that aims at improving the accessibility of residents to bus transit network.

(3) When $0 \leq \delta < w_0$, the new route should be common or branch. As shown in Fig. 10, when $0 \leq \delta < w_0$, $P(s)$ follows a generalized power-law distribution.



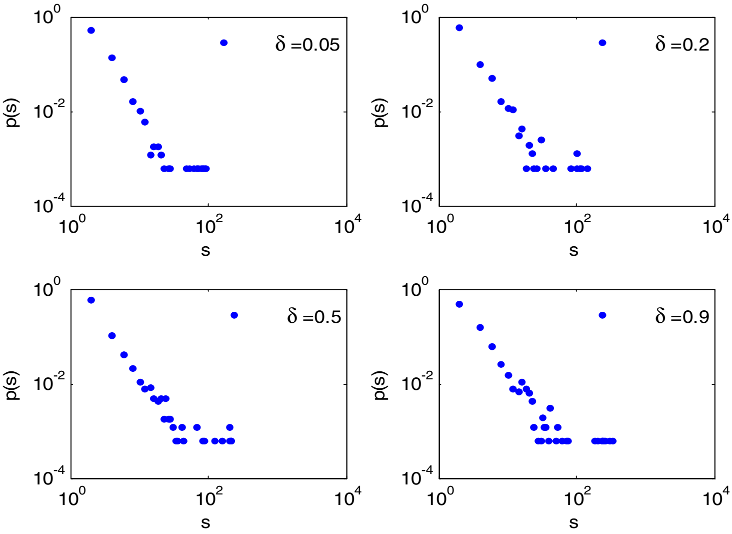
(a) When $p = 0.2$



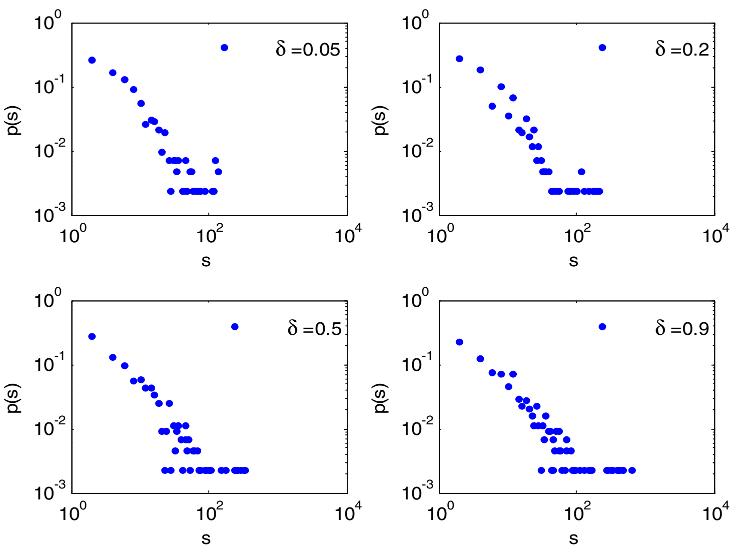
(b) When $p = 0.8$

Fig. 9. When $\delta \geq w_0$, the node's strength distributions $P(s)$ of network along the changes of perturbation δ on log–log scales.

When $p \to 1$ [Fig. 10(b)], the bowing phenomenon of distribution plot gradually appears with $\delta \to 0$, indicating that the network becomes much more homogeneous and there is a gradually transforming process from properties of scale-free to small-world, which is consistent with our theoretical result $\gamma \in [2\frac{1}{3}, +\infty)$. It



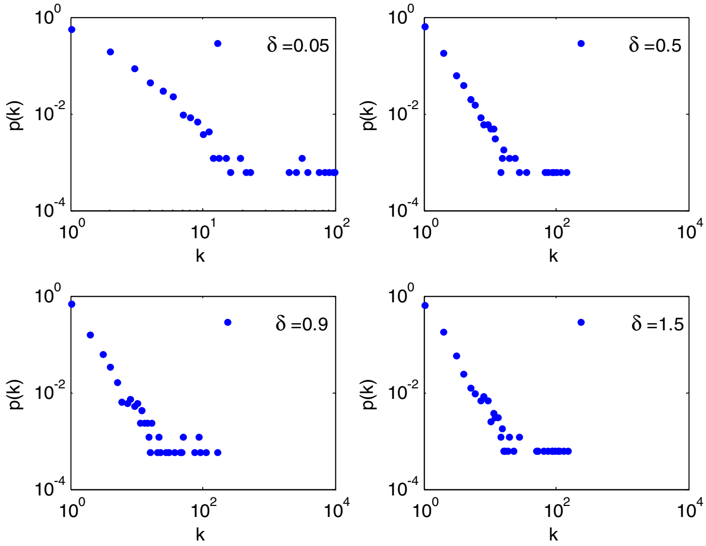
(a) When $p = 0.2$



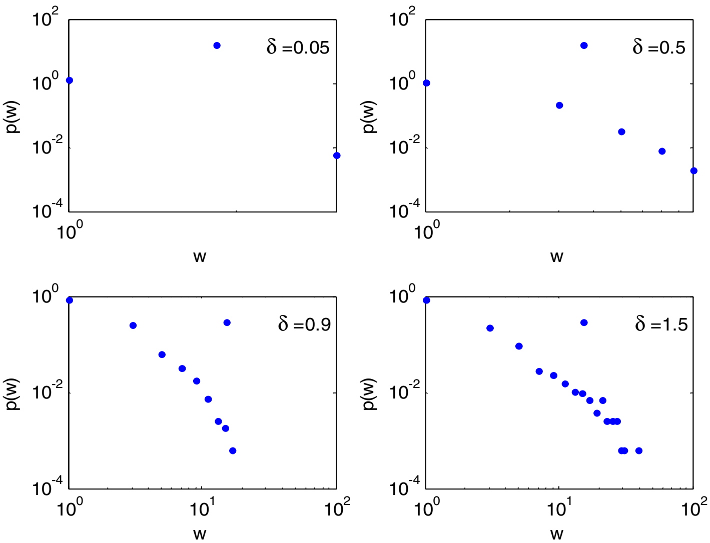
(b) When $p = 0.8$

Fig. 10. When $0 \leq \delta < w_0$, the node's strength distributions $P(s)$ of network along the changes of perturbation δ on log–log scales.

reveals that the new route does not generate great perturbation on the network, and most of attracted passengers can get to their travel destination without transfer, only a little flow needs to transfer to neighbors through terminal i or j. Thus, such routes are quite independent to the network and play a role to improve the density and homogeneity of network.



(a)



(b)

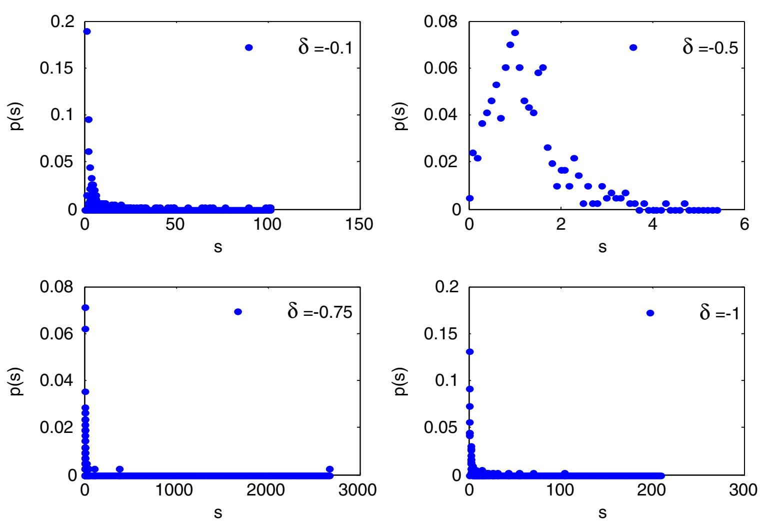
Fig. 11. When $p = 0.2, \delta > 0$, the node's degree and edge weight distributions along the changes of perturbation δ on a log–log scale. (a) The node's degree distribution $P(k)$. (b) The edge weight distributions $P(w)$.

In light of above analysis, we found that the probability P mainly impacts the exponent γ whereas it does not impacts network structure. Here, we just display the simulation results of degree distribution $P(k)$ and weight distribution $P(w)$ with $p = 0.2$. As shown in Fig. 11(a), it reveals that all $P(k)$ display the scale-free property with different δ, though with the increment of δ, it becomes more heterogeneous, according to the linear slopes that negative value is equal to the exponent γ. However, in Fig. 11(b), it's the increment of δ that makes $P(w)$ display more evident power-law behavior. It is because the neighboring perturbation flow triggered by a new route increases greatly, indicating that both the flow on the neighboring stations and routes are improved, consequently, the strength predominance of nodes is enhanced and then these nodes will be preferentially attached in next round of routes' generation; likewise, a new increase for these nodes' degree. It can explain the fact that there are some hub stations existing in transit networks.

In summary, when $\delta > 0$, with the increment of perturbation δ, not only the topology but also the flow distribution will exhibit the scale-free property with much more heterogeneity.

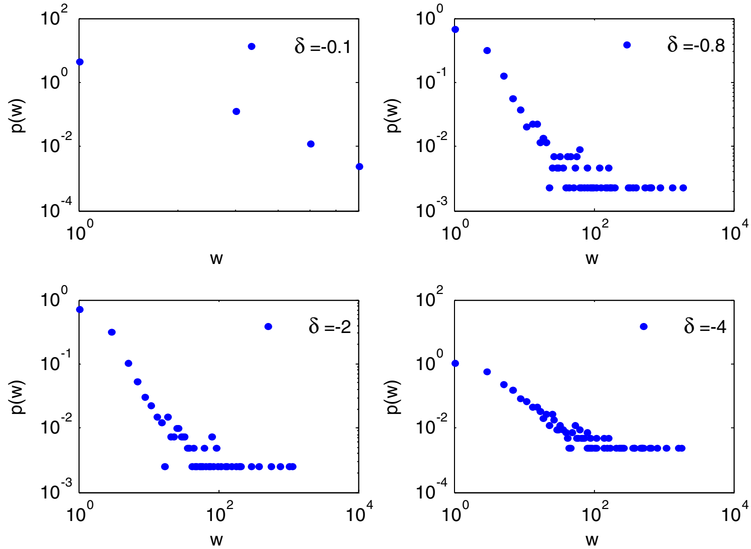
4.2. *When flow perturbation* $\delta < 0$

As $w_{ij}(0) \in (0,1)$ and $w_0 = 1$, firstly we simulate the case of $-w_0 \leq \delta < 0$. As shown in Fig. 12(a), when $-w_0 \leq \delta < 0$ and $\delta \neq -0.5w_0$, the long-tailed behavior of $P(s)$ shows that the typical scale-free property with $\gamma \in [2,3]$ exists in the network. It is because the decrement of neighboring flow is not larger than the additional increment of new route's flow, so the new route's nodes still keep the preferential characteristics. Moreover, when $\delta < -1$, especially when $\delta \to -\infty$, according to Eq. (10), if $p \to 1$, then $\gamma \to 3$; while $p \to 0$, $\gamma \to 2$, which indicates that $P(s)$ also follows a power-law distribution. But in real transit networks, the case of $\delta \to -\infty$



(a)

Fig. 12. When $p = 0.8$, $-w_0 \leq \delta < 0$, the node's strength and edge weight distributions along the changes of perturbation δ. (a) The node's strength distributions $P(s)$. (b) The edge weight distributions $P(w)$.



(b)

Fig. 12. (*Continued*)

is unlikely to occur, because a new route that will result in a great decrement of passenger flow in the whole network will not be allowed to open. Correspondingly, for the edge weight, when $\delta < 0$ and $\delta \to 0$ [shown in Fig. 12(b)], the heavy-tailed behavior of its distribution $p(w)$ shows more typical properties of scale-free.

Also, the simulation of degree distribution $P(k)$ (Fig. 13) shows that when $-w_0 < \delta < 0$ ($w_0 = 1$), there exist scale-free property for $P(k)$ with a heavy tail

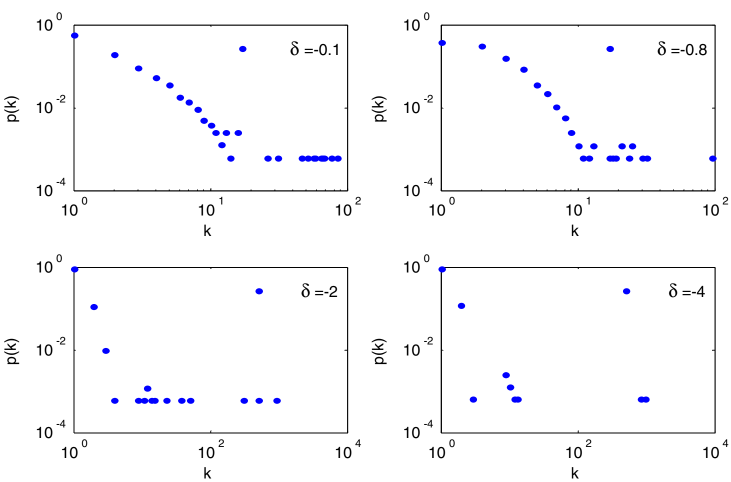


Fig. 13. When $p = 0.2$, $\delta < 0$, the degree distributions $P(k)$ of network along the changes of perturbation δ on log–log scales.

and nearly a line on a log–log scale. However, when $\delta < -w_0$, with decrement of δ, the degree distribution becomes more homogeneous and all nodes approximately possess the same degree. It reveals that when the perturbation δ triggers more passenger flow decreasing, the heterogeneity of topological network will be weakened after evolving.

To sum up, when $\delta < 0$, meaning that the open of a new route leads to the decrease of neighboring flow, it suggests that there may be some design flaws existing in the original network, for example, the great deviation of existing routes from the spatial linear path (the measured index is called nonlinear coefficient), then the new route is likely to make passengers minimize the transfers in neighboring routes. In this case, the open of new route is helpful to reduce the deviation or nonlinear coefficient, and improve the passengers' directness without transfer between key origin-destination pair. So passengers tend to take their trips on the new route rather than other existing routes, which is corresponding to decreasing flow on other routes.

5. Conclusion

In this paper, we present a comparative investigation on the Beijing's weighted TRN in R-space based on passenger flow to provide an empirical basis for exploring evolutional mechanism of TRNs. The power-law distributions both for degree and strength reveal that not only the topologic structure, bus also the flow distribution of Beijing's transit network in R-space are scale-free, even though the network is evolving. Also, their shorter average path length and small clustering coefficients further indicate that the TRNs exhibit scale-free property. Considering the topology and flow distribution coupling interaction, we propose an evolution model of weighted complex network for transit route system that is driven by passenger flow and flow perturbation triggered by topological change. The results obtained by theoretical analysis and numerical simulation have proven that our model can generate a TRN in R-space with scale-free properties for node's strength and degree distributions, which agrees well with our empirical results. With simulating the impacts of flow perturbation on the network's structure, we also can judge the function of new routes, which can be exploited to define the role of a route and the impact of passenger flow, so as to explore the countermeasures for routes' planning and adjustment in real transit networks much better.

However, this work is just a preliminary exploration of the evolutional mechanism for real networks. We simulated the evolutional process of transit networks from the view of macro-layout, whereas the real networks are complicated and exhibit much more diversity, in addition, the routing design and station locating are always coupled with passenger flow closely. Accordingly, more impact factors in the evolutional process should be further studied. Moreover, we regarded the new route's flow as a constant. The fact is that the flow is dynamic which may follow some kinds of distributions. The more realistic flow behavior and other mapping models for weighted transit networks should be explored to reflect the real networks' intrinsic properties more comprehensively.

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